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Double of boundary singularity of stable map from 3-manifold with boundary to 2-manifold

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1 Introduction

We consider singularities of the smooth map obtained as the "double" of a stable map from a 3-manifold with boundary to a 2-manifold without boundary. Still, we restrict our attention to local theory, and hence take a map between Euclidean spaces. Let (x, y, z) be a coordinate system of $\mathbb{R}^3$, let $\mathbb{R}^3_{\geq 0}$ denote the half space $\{z \geq 0\}$ in $\mathbb{R}^3$, and let $f\colon \mathbb{R}^3_{\geq 0} \to \mathbb{R}^2$ be a smooth map. By the "double" of f, we mean the map $F\colon \mathbb{R}^3 \to \mathbb{R}^2$ defined as $F(x, y, z) = f(x, y, z^2)$, which is clearly smooth. Note that, in the exterior of $\partial\mathbb{R}^3_{\geq 0}$, the transformation: $(x, y, z) \mapsto (x, y, z^2)$ is diffeomorphic at each point, and hence, the doubled map F inherits the types of singularities from the original map f. It might be naively hoped that, if a point p in $\partial\mathbb{R}^3_{\geq 0}$ is a stable boundary singular point of f, then p is a stable singular point of F. In this paper, we prove it for some types of stable boundary singular points, and disprove it for the other type.

Proposition 1. *With the above notation, we have the following.*

- *If p is a boundary regular point of f, then p is a regular point of F.*
- *If p is a boundary definite fold point of f, then p is a definite fold point of F.*
- *If p is a boundary indefinite fold point of f, then p is an indefinite fold point of F.*
- *If p is a boundary cusp point of f, then p is a cusp point of F.*
- *If p is a $\Sigma^{2,0}_{1,0}$ point of f, then p is an unstable singular point of F.*

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2 Preliminaries

In this section, we review standard definitions and facts with the following notation. Let M be a 3-dimensional C^∞ manifold possibly with boundary, N be a 2-dimensional

C^∞ manifold without boundary, and $f\colon M \to N$ be a C^∞ map. Let p be a point in M, and U be a sufficiently small neighborhood of p in M.

Singularity and boundary singularity are defined as follows. The point p is said to be a *regular point* of f if the differential $(df)_p : T_pM \to T_{f(p)}N$ is surjective, and a *singular point* of f otherwise. The point p is said to be a *boundary regular point* of f if $p \in \partial M$ and the differential $(d(f|_{\partial M}))_p : T_p\partial M \to T_{f(p)}N$ is surjective, and a *boundary singular point* of f otherwise. The set of singular points (resp. boundary singular points) of f is called the *singular set* (resp. the *boundary singular set*) of f, and denoted by $S(f)$ (resp. $S(f|_{\partial M})$).

Fold singularity is defined as follows. The point p is said to be a *fold point* of f if there are a local coordinate system (u, v, w) of M and one of N with respect to which $p = (0, 0, 0)$, $f(p) = (0, 0)$ and $f(u, v, w) = (u, v^2 + \varepsilon w^2)$ for $\varepsilon \in \{1, -1\}$. In particular, the fold point p is said to be *definite* if $\varepsilon = 1$, and *indefinite* if $\varepsilon = -1$. If p is an interior point of M, the singular set $S(f) \cap U$ is a regular arc which passes through p and consists only of fold points.

Cusp singularity is defined as follows. The point p is said to be a *cusp point* of f if there are a local coordinate system (u, v, w) of M and one of N with respect to which $p = (0, 0, 0)$, $f(p) = (0, 0)$ and $f(u, v, w) = (u, v^3 + uv + w^2)$. If p is an interior point of M, the singular set $S(f) \cap U$ is a regular arc which passes through p and consists only of fold points except for the cusp point p.

Boundary fold singularity is defined as follows. The point p is said to be a *boundary fold point* of f if there are a local coordinate system (u, v, w) of M and one of N with respect to which $p = (0, 0, 0)$, $f(p) = (0, 0)$, $M = \{w \geq 0\}$ and $f(u, v, w) = (u, v^2 + \varepsilon w)$ for $\varepsilon \in \{1, -1\}$. In particular, the boundary fold point p is said to be *definite* if $\varepsilon = 1$, and *indefinite* if $\varepsilon = -1$. Note that p is a boundary singular point but a regular point of f. The singular set $S(f) \cap U$ is empty, and the boundary singular set $S(f|_{\partial M}) \cap U$ is a regular arc which passes through p and consists only of boundary fold points.

Boundary cusp singularity is defined as follows. The point p is said to be a *boundary cusp point* of f if there are a local coordinate system (u, v, w) of M and one of N with respect to which $p = (0, 0, 0)$, $f(p) = (0, 0)$, $M = \{w \geq 0\}$ and $f(u, v, w) = (u, v^3 + uv + w)$. Note that p is a boundary singular point but a regular point of f. The singular set $S(f) \cap U$ is empty, and the boundary singular set $S(f|_{\partial M}) \cap U$ is the regular arc $\{3v^2 + u = w = 0\}$. This arc passes through p, and consists only of boundary fold points except for the boundary cusp point p.

$\Sigma^{2,0}_{1,0}$ singularity is defined as follows. The point p is said to be a $\Sigma^{2,0}_{1,0}$ *point* of f if there are a local coordinate system (u, v, w) of M and one of N with respect to which $p = (0, 0, 0)$, $f(p) = (0, 0)$, $M = \{w \geq 0\}$ and $f(u, v, w) = (u, v^2 + uw + \varepsilon w^2)$ for $\varepsilon \in \{1, -1\}$. Note that p is a boundary singular point and a fold point of f. The singular set $S(f) \cap U$ is a regular arc which has an endpoint at p and consists only of fold points. The boundary singular set $S(f|_{\partial M}) \cap U$ is a regular arc which passes through p and consists only of boundary fold points except for the $\Sigma^{2,0}_{1,0}$ point p.

The above singularities and boundary singularities are stable. Suppose that f is a stable map (see [1] for example). It is well known that any singular point of f is either a fold point or a cusp point. It follows from the results of Martins–Nabarro [2] and Shibata [5] that any boundary singular point of f is either a boundary fold point, a boundary cusp point, or a $\Sigma^{2,0}_{1,0}$ point. We refer the reader to [3] for more information.

Fold and cusp singularities can be recognized with the following criteria. Suppose that p is an interior point of M, and f has a local form: $f(x) = (f_1(x), f_2(x))$ for $x \in U$ such that $(df_1)_p \neq (0,0,0)$ and $(df_2)_p = (0,0,0)$. This implies that $\ker (df)_p$ has dimension 2. For C^∞ vector fields ξ_1 and ξ_2 on U, let $\mathbf{H}_{\xi_1,\xi_2} f_2$ denote the matrix

$$\begin{pmatrix} \xi_1\xi_1 f_2 & \xi_1\xi_2 f_2 \\ \xi_2\xi_1 f_2 & \xi_2\xi_2 f_2 \end{pmatrix}.$$

Provided that the vectors $(\xi_1)_p$ and $(\xi_2)_p$ are linearly independent, we regard $(\mathbf{H}_{\xi_1,\xi_2} f_2)_p$ as representing a linear transformation of $\left\langle (\xi_1)_p, (\xi_2)_p \right\rangle$ with respect to the basis $\left((\xi_1)_p, (\xi_2)_p \right)$. This allows us to treat $\ker (\mathbf{H}_{\xi_1,\xi_2} f_2)_p$ as a subspace of $\left\langle (\xi_1)_p, (\xi_2)_p \right\rangle$. Saji [4] gave criteria for recognizing general Morin singularities, and the following are those in special cases.

Theorem 2 (Saji). *The point p is a fold point of f if there exist C^∞ vector fields η_1 and η_2 on U such that*

- $\ker (df)_p = \left\langle (\eta_1)_p, (\eta_2)_p \right\rangle$,
- $\ker (\mathbf{H}_{\eta_1,\eta_2} f_2)_p = \{\mathbf{0}\}$.

Moreover, the fold point p is definite (resp. indefinite) if $(\mathbf{H}_{\eta_1,\eta_2} f_2)_p$ has eigenvalues of definite (resp. indefinite) sign.

Theorem 3 (Saji). *The point p is a cusp point of f if there exist C^∞ vector fields η_1 and η_2 on U such that*

- $\ker (df)_p = \left\langle (\eta_1)_p, (\eta_2)_p \right\rangle$,
- $(\eta_1)_q \in \ker (df)_q$ *for* $q \in S(f) \cap U$,
- $\ker (\mathbf{H}_{\eta_1,\eta_2} f_2)_p = \left\langle (\eta_1)_p \right\rangle$,
- $(d(\eta_1 f_2))_p \neq (0,0,0)$,
- $(\eta_1\eta_1\eta_1 f_2)_p \neq 0$.

3 Proof

In this section, we give a proof of Proposition 1. We use the notation of Introduction.

3.1 Regular case

The first assertion of the proposition can be proved almost immediately as follows. Suppose that p is a boundary regular point of f. By the definition, the differential $\left(d \left(f|_{\partial \mathbb{R}^3_{\geq 0}} \right) \right)_p$ is surjective. Since $\partial \mathbb{R}^3_{\geq 0} = \{z = 0\}$ and $F(x,y,z) = f(x,y,z^2)$, the maps f and F coincide in $\partial \mathbb{R}^3_{\geq 0}$, and hence $\left(d \left(F|_{\partial \mathbb{R}^3_{\geq 0}} \right) \right)_p$ is surjective. It implies that $(dF)_p$ is also surjective. Thus, p is a regular point of F.

3.2 Fold case

In this subsection, we give proofs of the second and third assertions of the proposition. Suppose that p is a boundary fold point of f.

The original map and the doubled map have local forms as follows. On one hand, there are local coordinate systems (u,v,w) and (s,t) of $\mathbb{R}^3$ and $\mathbb{R}^2$, respectively, with respect to which $p=(0,0,0)$, $f(p)=(0,0)$, $\mathbb{R}^3_{\geq 0}=\{w\geq 0\}$ and $f(u,v,w)=(u,v^2+\varepsilon w)$, where $\varepsilon=1$ if the boundary fold point p is definite and $\varepsilon=-1$ if indefinite. On the other hand, $F(x,y,z)=f\left(x,y,z^2\right)$ with respect to the coordinate system (x,y,z) of $\mathbb{R}^3$. We may suppose that $p=(0,0,0)$ with respect to (x,y,z). Suppose that F has a local form: $F(x,y,z)=(F_1(x,y,z),F_2(x,y,z))$ with respect to (x,y,z) and (s,t).

The relevant coordinate systems are related as follows. There is a coordinate transformation: $(x,y,z)\mapsto(u(x,y,z),v(x,y,z),w(x,y,z))$. Since $\{z\geq 0\}=\{w\geq 0\}$ and $p\in\{z=0\}=\{w=0\}$, the transformation satisfies the conditions that

$$\left(\frac{\partial u}{\partial x}\right)_p\left(\frac{\partial v}{\partial y}\right)_p-\left(\frac{\partial u}{\partial y}\right)_p\left(\frac{\partial v}{\partial x}\right)_p\neq 0,$$

$$\left(\frac{\partial w}{\partial x}\right)_p=\left(\frac{\partial w}{\partial y}\right)_p=\left(\frac{\partial^2 w}{\partial x^2}\right)_p=\left(\frac{\partial^2 w}{\partial y^2}\right)_p=\left(\frac{\partial^2 w}{\partial x\partial y}\right)_p=0,$$

$$\left(\frac{\partial w}{\partial z}\right)_p>0.$$

In particular, the top inequality implies that

$$\left(\left(\frac{\partial u}{\partial x}\right)_p,\left(\frac{\partial u}{\partial y}\right)_p\right)\neq(0,0).$$

We calculate partial derivatives with respect to the coordinates as follows. Note that F_1 and F_2 have the local forms: $F_1(x,y,z)=u\left(x,y,z^2\right)$ and $F_2(x,y,z)=\left(v^2+\varepsilon w\right)\left(x,y,z^2\right)$, respectively, under the coordinate transformation: $(x,y,z)\mapsto(u,v,w)$. By the chain rule, for example,

$$\begin{aligned}
&\frac{\partial F_2}{\partial z}(x,y,z)\\
&=\frac{\partial}{\partial z}\left(\left(v^2+\varepsilon w\right)\left(x,y,z^2\right)\right)\\
&=\left(\frac{\partial}{\partial z}x\right)\left(\left(\frac{\partial}{\partial x}\left(v^2+\varepsilon w\right)\right)\left(x,y,z^2\right)\right)+\left(\frac{\partial}{\partial z}y\right)\left(\left(\frac{\partial}{\partial y}\left(v^2+\varepsilon w\right)\right)\left(x,y,z^2\right)\right)\\
&\quad+\left(\frac{\partial}{\partial z}z^2\right)\left(\left(\frac{\partial}{\partial z}\left(v^2+\varepsilon w\right)\right)\left(x,y,z^2\right)\right)\\
&=2z\left(\left(\frac{\partial}{\partial z}\left(v^2+\varepsilon w\right)\right)\left(x,y,z^2\right)\right)\\
&=2z\left(\left(2v\frac{\partial v}{\partial z}+\varepsilon\frac{\partial w}{\partial z}\right)\left(x,y,z^2\right)\right),
\end{aligned}$$

$$
\begin{aligned}
&\frac{\partial^2 F_2}{\partial z^2}(x,y,z)\\
&= \frac{\partial}{\partial z}\left(2z\left(\left(2v\frac{\partial v}{\partial z}+\varepsilon\frac{\partial w}{\partial z}\right)(x,y,z^2)\right)\right)\\
&= 2\left(2v\frac{\partial v}{\partial z}+\varepsilon\frac{\partial w}{\partial z}\right)(x,y,z^2)+2z\frac{\partial}{\partial z}\left(\left(2v\frac{\partial v}{\partial z}+\varepsilon\frac{\partial w}{\partial z}\right)(x,y,z^2)\right)\\
&= 2\left(2v\frac{\partial v}{\partial z}+\varepsilon\frac{\partial w}{\partial z}\right)(x,y,z^2)+4z^2\left(\left(\frac{\partial}{\partial z}\left(2v\frac{\partial v}{\partial z}+\varepsilon\frac{\partial w}{\partial z}\right)\right)(x,y,z^2)\right)\\
&= 2\left(2v\frac{\partial v}{\partial z}+\varepsilon\frac{\partial w}{\partial z}\right)(x,y,z^2)+4z^2\left(\left(2\left(\frac{\partial v}{\partial z}\right)^2+2v\frac{\partial^2 v}{\partial z^2}+\varepsilon\frac{\partial^2 w}{\partial z^2}\right)(x,y,z^2)\right),
\end{aligned}
$$

and similarly,

$$
\begin{aligned}
\frac{\partial F_1}{\partial x}(x,y,z) &= \frac{\partial u}{\partial x}(x,y,z^2),\\
\frac{\partial F_1}{\partial y}(x,y,z) &= \frac{\partial u}{\partial y}(x,y,z^2),\\
\frac{\partial F_1}{\partial z}(x,y,z) &= 2z\left(\frac{\partial u}{\partial z}(x,y,z^2)\right),\\
\frac{\partial F_2}{\partial x}(x,y,z) &= \left(2v\frac{\partial v}{\partial x}+\varepsilon\frac{\partial w}{\partial x}\right)(x,y,z^2),\\
\frac{\partial F_2}{\partial y}(x,y,z) &= \left(2v\frac{\partial v}{\partial y}+\varepsilon\frac{\partial w}{\partial y}\right)(x,y,z^2),\\
\frac{\partial^2 F_2}{\partial x^2}(x,y,z) &= \left(2\left(\frac{\partial v}{\partial x}\right)^2+2v\frac{\partial^2 v}{\partial x^2}+\varepsilon\frac{\partial^2 w}{\partial x^2}\right)(x,y,z^2),\\
\frac{\partial^2 F_2}{\partial y^2}(x,y,z) &= \left(2\left(\frac{\partial v}{\partial y}\right)^2+2v\frac{\partial^2 v}{\partial y^2}+\varepsilon\frac{\partial^2 w}{\partial y^2}\right)(x,y,z^2),\\
\frac{\partial^2 F_2}{\partial x\partial y}(x,y,z) &= \left(2\frac{\partial v}{\partial x}\frac{\partial v}{\partial x}+2v\frac{\partial^2 v}{\partial x\partial y}+\varepsilon\frac{\partial^2 w}{\partial x\partial y}\right)(z,y,z^2),\\
\frac{\partial^2 F_2}{\partial x\partial z}(x,y,z) &= 2z\left(\left(2\frac{\partial v}{\partial x}\frac{\partial v}{\partial z}+2v\frac{\partial^2 v}{\partial x\partial z}+\varepsilon\frac{\partial^2 w}{\partial x\partial z}\right)(x,y,z^2)\right),\\
\frac{\partial^2 F_2}{\partial y\partial z}(x,y,z) &= 2z\left(\left(2\frac{\partial v}{\partial y}\frac{\partial v}{\partial z}+2v\frac{\partial^2 v}{\partial y\partial z}+\varepsilon\frac{\partial^2 w}{\partial y\partial z}\right)(x,y,z^2)\right).
\end{aligned}
$$

Since $p=(0,0,0)$ with respect to both (x,y,z) and (u,v,w), for example,

$$
\begin{aligned}
\left(\frac{\partial^2 F_2}{\partial x^2}\right)_p &= \left(2\left(\frac{\partial v}{\partial x}\right)^2+2v\frac{\partial^2 v}{\partial x^2}+\varepsilon\frac{\partial^2 w}{\partial x^2}\right)(0,0,0^2)\\
&= 2\left(\frac{\partial v}{\partial x}\right)_p^2+2\cdot 0\left(\frac{\partial^2 v}{\partial x^2}\right)_p+\varepsilon\left(\frac{\partial^2 w}{\partial x^2}\right)_p=2\left(\frac{\partial v}{\partial x}\right)_p^2,
\end{aligned}
$$

and similarly,

$$\left(\frac{\partial F_1}{\partial x}\right)_p = \left(\frac{\partial u}{\partial x}\right)_p,$$
$$\left(\frac{\partial F_1}{\partial y}\right)_p = \left(\frac{\partial u}{\partial y}\right)_p,$$
$$\left(\frac{\partial^2 F_2}{\partial y^2}\right)_p = 2\left(\frac{\partial v}{\partial y}\right)_p^2,$$
$$\left(\frac{\partial^2 F_2}{\partial z^2}\right)_p = 2\varepsilon\left(\frac{\partial w}{\partial z}\right)_p,$$
$$\left(\frac{\partial^2 F_2}{\partial x\partial y}\right)_p = 2\left(\frac{\partial v}{\partial x}\right)_p\left(\frac{\partial v}{\partial y}\right)_p,$$
$$\left(\frac{\partial F_1}{\partial z}\right)_p = \left(\frac{\partial F_2}{\partial x}\right)_p = \left(\frac{\partial F_2}{\partial y}\right)_p = \left(\frac{\partial F_2}{\partial z}\right)_p = \left(\frac{\partial^2 F_2}{\partial x\partial z}\right)_p = \left(\frac{\partial^2 F_2}{\partial y\partial z}\right)_p = 0.$$

We choose a pair of vector fields and calculate derivatives with respect to them as follows. Let η_1 and η_2 be C^∞ vector fields on U as

$$\eta_1 = \left(\frac{\partial u}{\partial y}\right)_p \frac{\partial}{\partial x} - \left(\frac{\partial u}{\partial x}\right)_p \frac{\partial}{\partial y},$$
$$\eta_2 = \frac{\partial}{\partial z}.$$

Noting that the coefficients of η_1 are constants,

$$\eta_1 F_2 = \left(\frac{\partial u}{\partial y}\right)_p \frac{\partial F_2}{\partial x} - \left(\frac{\partial u}{\partial x}\right)_p \frac{\partial F_2}{\partial y},$$
$$\begin{aligned}\eta_1\eta_1 F_2 &= \left(\left(\frac{\partial u}{\partial y}\right)_p \frac{\partial}{\partial x} - \left(\frac{\partial u}{\partial x}\right)_p \frac{\partial}{\partial y}\right)\left(\left(\frac{\partial u}{\partial y}\right)_p \frac{\partial F_2}{\partial x} - \left(\frac{\partial u}{\partial x}\right)_p \frac{\partial F_2}{\partial y}\right)\\ &= \left(\frac{\partial u}{\partial y}\right)_p^2 \frac{\partial^2 F_2}{\partial x^2} - 2\left(\frac{\partial u}{\partial x}\right)_p\left(\frac{\partial u}{\partial y}\right)_p \frac{\partial^2 F_2}{\partial x\partial y} + \left(\frac{\partial u}{\partial x}\right)_p^2 \frac{\partial^2 F_2}{\partial y^2}.\end{aligned}$$

By the results of the previous paragraph,

$$(\eta_1 F_2)_p = \left(\frac{\partial u}{\partial y}\right)_p\left(\frac{\partial F_2}{\partial x}\right)_p - \left(\frac{\partial u}{\partial x}\right)_p\left(\frac{\partial F_2}{\partial y}\right)_p = 0,$$
$$\begin{aligned}(\eta_1\eta_1 F_2)_p &= \left(\frac{\partial u}{\partial y}\right)_p^2\left(\frac{\partial^2 F_2}{\partial x^2}\right)_p - 2\left(\frac{\partial u}{\partial x}\right)_p\left(\frac{\partial u}{\partial y}\right)_p\left(\frac{\partial^2 F_2}{\partial x\partial y}\right)_p + \left(\frac{\partial u}{\partial x}\right)_p^2\left(\frac{\partial^2 F_2}{\partial y^2}\right)_p\\ &= 2\left(\frac{\partial u}{\partial y}\right)_p^2\left(\frac{\partial v}{\partial x}\right)_p^2 - 4\left(\frac{\partial u}{\partial x}\right)_p\left(\frac{\partial u}{\partial y}\right)_p\left(\frac{\partial v}{\partial x}\right)_p\left(\frac{\partial v}{\partial y}\right)_p + 2\left(\frac{\partial u}{\partial x}\right)_p^2\left(\frac{\partial v}{\partial y}\right)_p^2\\ &= 2\left(\left(\frac{\partial u}{\partial x}\right)_p\left(\frac{\partial v}{\partial y}\right)_p - \left(\frac{\partial u}{\partial y}\right)_p\left(\frac{\partial v}{\partial x}\right)_p\right)^2 > 0.\end{aligned}$$

Similarly, we can obtain that $(\eta_1 F_1)_p = (\eta_2 F_1)_p = (\eta_2 F_2)_p = (\eta_1\eta_2 F_2)_p = (\eta_2\eta_1 F_2)_p = 0$ and

$$(\eta_2\eta_2 F_2)_p = \left(\frac{\partial^2 F_2}{\partial z^2}\right)_p = 2\varepsilon\left(\frac{\partial w}{\partial z}\right)_p \neq 0.$$

We are now ready to complete the proofs. By the above,

$$(dF_1)_p = \left(\left(\frac{\partial F_1}{\partial x}\right)_p, \left(\frac{\partial F_1}{\partial y}\right)_p, \left(\frac{\partial F_1}{\partial z}\right)_p\right) = \left(\left(\frac{\partial u}{\partial x}\right)_p, \left(\frac{\partial u}{\partial y}\right)_p, 0\right) \neq (0,0,0),$$

$$(dF_2)_p = \left(\left(\frac{\partial F_2}{\partial x}\right)_p, \left(\frac{\partial F_2}{\partial y}\right)_p, \left(\frac{\partial F_2}{\partial z}\right)_p\right) = (0,0,0).$$

Since $(\eta_1)_p$ and $(\eta_2)_p$ are linearly independent, and $(\eta_1 F_1)_p = (\eta_2 F_1)_p = (\eta_1 F_2)_p = (\eta_2 F_2)_p = 0$, we obtain the condition that $\ker(dF)_p = \left\langle (\eta_1)_p, (\eta_2)_p \right\rangle$. The matrix

$$\begin{pmatrix} (\eta_1\eta_1 F_2)_p & (\eta_1\eta_2 F_2)_p \\ (\eta_2\eta_1 F_2)_p & (\eta_2\eta_2 F_2)_p \end{pmatrix},$$

denoted by $(\mathbf{H}_{\eta_1,\eta_2} F_2)_p$, is equal to

$$\begin{pmatrix} 2\left(\left(\frac{\partial u}{\partial x}\right)_p\left(\frac{\partial v}{\partial y}\right)_p - \left(\frac{\partial u}{\partial y}\right)_p\left(\frac{\partial v}{\partial x}\right)_p\right)^2 & 0 \\ 0 & 2\varepsilon\left(\frac{\partial w}{\partial z}\right)_p \end{pmatrix},$$

which shows that $\ker(\mathbf{H}_{\eta_1,\eta_2} F_2)_p = \{\mathbf{0}\}$. By Theorem 2, the point p is a fold point of F. Moreover, the fold point p of F is definite (resp. indefinite) if $\varepsilon > 0$ (resp. $\varepsilon < 0$), that is to say, the boundary fold point p of f is definite (resp. indefinite).

3.3 Cusp case

In this subsection, we give a proof of the fourth assertion of the proposition. Suppose that p is a boundary cusp point of f. Let $S(F)$ denote the singular set of F, let U be a sufficiently small neighborhood of p in $\mathbb{R}^3$, and let q be any point in $S(F) \cap U$.

The original map and the doubled map have local forms as follows. On one hand, there are local coordinate systems (u, v, w) and (s, t) of $\mathbb{R}^3$ and $\mathbb{R}^2$, respectively, with respect to which $p = (0,0,0)$, $f(p) = (0,0)$, $\mathbb{R}^3_{\geq 0} = \{w \geq 0\}$ and $f(u,v,w) = (u, v^3 + uv + w)$. On the other hand, $F(x,y,z) = f(x,y,z^2)$ with respect to the coordinate system (x,y,z) of $\mathbb{R}^3$. We may suppose that $p = (0,0,0)$ with respect to (x,y,z). Suppose that F has a local form: $F(x,y,z) = (F_1(x,y,z), F_2(x,y,z))$ with respect to (x,y,z) and (s,t).

We detect the singular set of the doubled map as follows. Recall that the original map f has no singular points in U. The doubled map F inherits the regularity of f in $U \setminus \partial\mathbb{R}^3_{\geq 0}$. Recall also that f has only boundary regular points in $\partial\mathbb{R}^3_{\geq 0} \setminus \{3v^2 + u = w = 0\}$, and only boundary fold points in $\{3v^2 + u = w = 0\} \setminus \{p\}$. By the results of the previous subsections, $S(F)\cap U$ is either $\{3v^2+u=w=0\}\setminus\{p\}$ or $\{3v^2+u=w=0\}$. Since the singular set is a closed set in general, we conclude that $S(F)\cap U=\{3v^2+u=w=0\}$. Hence q is possibly p.

The relevant coordinate systems are related as follows. There is a coordinate transformation: $(x,y,z)\mapsto(u(x,y,z),v(x,y,z),w(x,y,z))$. Since $\{z\geq 0\}=\{w\geq 0\}$ and $q\in\{z=0\}=\{w=0\}$, the transformation satisfies the conditions that

$$\left(\frac{\partial u}{\partial x}\right)_q\left(\frac{\partial v}{\partial y}\right)_q-\left(\frac{\partial u}{\partial y}\right)_q\left(\frac{\partial v}{\partial x}\right)_q\neq 0,$$

$$\left(\frac{\partial w}{\partial x}\right)_q=\left(\frac{\partial w}{\partial y}\right)_q=\left(\frac{\partial^2 w}{\partial x^2}\right)_q=\left(\frac{\partial^2 w}{\partial y^2}\right)_q=\left(\frac{\partial^2 w}{\partial x\partial y}\right)_q=0,$$

$$\left(\frac{\partial w}{\partial z}\right)_q>0.$$

In particular, the top inequality implies that

$$\left(\left(\frac{\partial u}{\partial x}\right)_q,\left(\frac{\partial u}{\partial y}\right)_q\right)\neq(0,0).$$

We calculate partial derivatives with respect to the coordinates similarly to those in the previous subsection. We can obtain that

$$\frac{\partial F_1}{\partial x}(x,y,z)=\frac{\partial u}{\partial x}\left(x,y,z^2\right),$$

$$\frac{\partial F_1}{\partial y}(x,y,z)=\frac{\partial u}{\partial y}\left(x,y,z^2\right),$$

$$\frac{\partial F_1}{\partial z}(x,y,z)=2z\left(\frac{\partial u}{\partial z}\left(x,y,z^2\right)\right),$$

$$\frac{\partial F_2}{\partial x}(x,y,z)=\left(v\frac{\partial u}{\partial x}+\left(3v^2+u\right)\frac{\partial v}{\partial x}+\frac{\partial w}{\partial x}\right)\left(x,y,z^2\right),$$

$$\frac{\partial F_2}{\partial y}(x,y,z)=\left(v\frac{\partial u}{\partial y}+\left(3v^2+u\right)\frac{\partial v}{\partial y}+\frac{\partial w}{\partial y}\right)\left(x,y,z^2\right),$$

$$\frac{\partial F_2}{\partial z}(x,y,z)=2z\left(\left(v\frac{\partial u}{\partial z}+\left(3v^2+u\right)\frac{\partial v}{\partial z}+\frac{\partial w}{\partial z}\right)\left(x,y,z^2\right)\right),$$

$$\frac{\partial^2 F_2}{\partial x^2}(x,y,z)=\left(2\frac{\partial u}{\partial x}\frac{\partial v}{\partial x}+v\frac{\partial^2 u}{\partial x^2}+6v\left(\frac{\partial v}{\partial x}\right)^2+\left(3v^2+u\right)\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 w}{\partial x^2}\right)\left(x,y,z^2\right),$$

$$\frac{\partial^2 F_2}{\partial y^2}(x,y,z)=\left(2\frac{\partial u}{\partial y}\frac{\partial v}{\partial y}+v\frac{\partial^2 u}{\partial y^2}+6v\left(\frac{\partial v}{\partial y}\right)^2+\left(3v^2+u\right)\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 w}{\partial y^2}\right)\left(x,y,z^2\right),$$

$$\begin{aligned}\frac{\partial^2 F_2}{\partial z^2}(x,y,z)=&\,2\left(v\frac{\partial u}{\partial z}+(3v^2+u)\frac{\partial v}{\partial z}+\frac{\partial w}{\partial z}\right)\left(x,y,z^2\right)\\&+4z^2\left(\left(\frac{\partial v}{\partial z}\frac{\partial u}{\partial z}+v\frac{\partial^2 u}{\partial z^2}+\left(6v\frac{\partial v}{\partial z}+\frac{\partial u}{\partial z}\right)\frac{\partial v}{\partial z}\right.\right.\\&\left.\left.+(3v^2+u)\frac{\partial^2 v}{\partial z^2}+\frac{\partial^2 w}{\partial z^2}\right)\left(x,y,z^2\right)\right),\end{aligned}$$

$$\frac{\partial^2 F_2}{\partial x \partial y}(x,y,z) = \left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial x} + v\frac{\partial^2 u}{\partial x \partial y} + 6v\frac{\partial v}{\partial x}\frac{\partial v}{\partial y} \right.$$
$$\left. + \left(3v^2 + u\right)\frac{\partial^2 v}{\partial x \partial y} + \frac{\partial^2 w}{\partial x \partial y}\right)\left(x, y, z^2\right),$$
$$\frac{\partial^2 F_2}{\partial x \partial z}(x,y,z) = 2z\left(\left(\frac{\partial v}{\partial x}\frac{\partial u}{\partial z} + v\frac{\partial^2 u}{\partial x \partial z} + \left(6v\frac{\partial v}{\partial x} + \frac{\partial u}{\partial x}\right)\frac{\partial v}{\partial z}\right.\right.$$
$$\left.\left. +(3v^2+u)\frac{\partial^2 v}{\partial x \partial z} + \frac{\partial^2 w}{\partial x \partial z}\right)(u,v,w^2)\right),$$
$$\frac{\partial^2 F_2}{\partial y \partial z}(x,y,z) = 2z\left(\left(\frac{\partial v}{\partial y}\frac{\partial u}{\partial z} + v\frac{\partial^2 u}{\partial y \partial z} + \left(6v\frac{\partial v}{\partial y} + \frac{\partial u}{\partial y}\right)\frac{\partial v}{\partial z}\right.\right.$$
$$\left.\left. +(3v^2+u)\frac{\partial^2 v}{\partial y \partial z} + \frac{\partial^2 w}{\partial y \partial z}\right)(u,v,w^2)\right),$$
$$\frac{\partial^3 F_2}{\partial x^3} = \left(3\frac{\partial u}{\partial x}\frac{\partial^2 v}{\partial x^2} + 6\left(\frac{\partial v}{\partial x}\right)^3 + 3\frac{\partial v}{\partial x}\frac{\partial^2 u}{\partial x^2} + 18v\frac{\partial v}{\partial x}\frac{\partial^2 v}{\partial x^2}\right.$$
$$\left. +v\frac{\partial^3 u}{\partial x^3} + (3v^2+u)\frac{\partial^3 v}{\partial x^3} + \frac{\partial^3 w}{\partial x^3}\right)(x,y,z^2),$$
$$\frac{\partial^3 F_2}{\partial x^2 \partial y} = \left(2\frac{\partial u}{\partial x}\frac{\partial^2 v}{\partial x \partial y} + \frac{\partial u}{\partial y}\frac{\partial^2 v}{\partial x^2} + 6\left(\frac{\partial v}{\partial x}\right)^2\frac{\partial v}{\partial y} + 2\frac{\partial v}{\partial x}\frac{\partial^2 u}{\partial x \partial y} + 12v\frac{\partial v}{\partial x}\frac{\partial^2 v}{\partial x \partial y}\right.$$
$$\left. +\frac{\partial v}{\partial y}\frac{\partial^2 u}{\partial x^2} + 6v\frac{\partial v}{\partial y}\frac{\partial^2 v}{\partial x^2} + v\frac{\partial^3 u}{\partial x^2 \partial y} + (3v^2+u)\frac{\partial^3 v}{\partial x^2 \partial y} + \frac{\partial^3 w}{\partial x^2 \partial y}\right)(x,y,z^2),$$
$$\frac{\partial^3 F_2}{\partial x \partial y^2} = \left(\frac{\partial u}{\partial x}\frac{\partial^2 v}{\partial y^2} + 2\frac{\partial u}{\partial y}\frac{\partial^2 v}{\partial x \partial y} + 6\frac{\partial v}{\partial x}\left(\frac{\partial v}{\partial y}\right)^2 + \frac{\partial v}{\partial x}\frac{\partial^2 u}{\partial y^2} + 6v\frac{\partial v}{\partial x}\frac{\partial^2 v}{\partial y^2}\right.$$
$$\left. +2\frac{\partial v}{\partial y}\frac{\partial^2 u}{\partial x \partial y} + 12v\frac{\partial v}{\partial y}\frac{\partial^2 v}{\partial x \partial y} + v\frac{\partial^3 u}{\partial x \partial y^2} + (3v^2+u)\frac{\partial^3 v}{\partial x \partial y^2} + \frac{\partial^3 w}{\partial x \partial y^2}\right)(x,y,z^2),$$
$$\frac{\partial^3 F_2}{\partial y^3} = \left(3\frac{\partial u}{\partial y}\frac{\partial^2 v}{\partial y^2} + 6\left(\frac{\partial v}{\partial y}\right)^3 + 3\frac{\partial v}{\partial y}\frac{\partial^2 u}{\partial y^2} + 18v\frac{\partial v}{\partial y}\frac{\partial^2 v}{\partial y^2}\right.$$
$$\left. +v\frac{\partial^3 u}{\partial y^3} + (3v^2+u)\frac{\partial^3 v}{\partial y^3} + \frac{\partial^3 w}{\partial y^3}\right)(x,y,z^2).$$

Let $(x_q, y_q, 0)$ and $(u_q, v_q, 0)$ be the coordinate representations of q with respect to the coordinate systems (x, y, z) and (u, v, w), respectively. Noting that $3v_q^2 + u_q = 0$, for example,

$$\left(\frac{\partial F_1}{\partial x}\right)_q = \frac{\partial u}{\partial x}(x_q, y_q, 0^2) = \left(\frac{\partial u}{\partial x}\right)_q,$$
$$\left(\frac{\partial F_2}{\partial x}\right)_q = \left(v\frac{\partial u}{\partial x} + \left(3v^2+u\right)\frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}\right)\left(x_q, y_q, 0^2\right)$$
$$= v_q\left(\frac{\partial u}{\partial x}\right)_q + (3v_q^2+u_q)\left(\frac{\partial v}{\partial x}\right)_q + \left(\frac{\partial w}{\partial y}\right)_q = v_q\left(\frac{\partial u}{\partial x}\right)_q,$$

and similarly,

$$\left(\frac{\partial F_1}{\partial y}\right)_q = \left(\frac{\partial u}{\partial y}\right)_q,$$

$$\left(\frac{\partial F_2}{\partial y}\right)_q = v_q\left(\frac{\partial u}{\partial y}\right)_q.$$

Noting that $v_p = 0$,

$$\left(\frac{\partial F_2}{\partial x}\right)_p = \left(\frac{\partial F_2}{\partial y}\right)_p = 0.$$

Similarly, we can obtain that

$$\left(\frac{\partial^2 F_2}{\partial x^2}\right)_p = 2\left(\frac{\partial u}{\partial x}\right)_p\left(\frac{\partial v}{\partial x}\right)_p,$$

$$\left(\frac{\partial^2 F_2}{\partial y^2}\right)_p = 2\left(\frac{\partial u}{\partial y}\right)_p\left(\frac{\partial v}{\partial y}\right)_p,$$

$$\left(\frac{\partial^2 F_2}{\partial z^2}\right)_p = 2\left(\frac{\partial w}{\partial z}\right)_p,$$

$$\left(\frac{\partial^2 F_2}{\partial x\partial y}\right)_p = \left(\frac{\partial u}{\partial x}\right)_p\left(\frac{\partial v}{\partial y}\right)_p + \left(\frac{\partial u}{\partial y}\right)_p\left(\frac{\partial v}{\partial x}\right)_p,$$

$$\left(\frac{\partial^3 F_2}{\partial x^3}\right)_p = 3\left(\frac{\partial u}{\partial x}\right)_p\left(\frac{\partial^2 v}{\partial x^2}\right)_p + 6\left(\frac{\partial v}{\partial x}\right)_p^3 + 3\left(\frac{\partial v}{\partial x}\right)_p\left(\frac{\partial^2 u}{\partial x^2}\right)_p,$$

$$\begin{aligned}\left(\frac{\partial^3 F_2}{\partial x^2\partial y}\right)_p = {} & 2\left(\frac{\partial u}{\partial x}\right)_p\left(\frac{\partial^2 v}{\partial x\partial y}\right)_p + \left(\frac{\partial u}{\partial y}\right)_p\left(\frac{\partial^2 v}{\partial x^2}\right)_p + 6\left(\frac{\partial v}{\partial x}\right)_p^2\left(\frac{\partial v}{\partial y}\right)_p \\ & + 2\left(\frac{\partial v}{\partial x}\right)_p\left(\frac{\partial^2 u}{\partial x\partial y}\right)_p + \left(\frac{\partial v}{\partial y}\right)_p\left(\frac{\partial^2 u}{\partial x^2}\right)_p,\end{aligned}$$

$$\begin{aligned}\left(\frac{\partial^3 F_2}{\partial x\partial y^2}\right)_p = {} & \left(\frac{\partial u}{\partial x}\right)_p\left(\frac{\partial^2 v}{\partial y^2}\right)_p + 2\left(\frac{\partial u}{\partial y}\right)_p\left(\frac{\partial^2 v}{\partial x\partial y}\right)_p + 6\left(\frac{\partial v}{\partial x}\right)_p\left(\frac{\partial v}{\partial y}\right)_p^2 \\ & + \left(\frac{\partial v}{\partial x}\right)_p\left(\frac{\partial^2 u}{\partial y^2}\right)_p + 2\left(\frac{\partial v}{\partial y}\right)_p\left(\frac{\partial^2 u}{\partial x\partial y}\right)_p,\end{aligned}$$

$$\left(\frac{\partial^3 F_2}{\partial y^3}\right)_p = 3\left(\frac{\partial u}{\partial y}\right)_p\left(\frac{\partial^2 v}{\partial y^2}\right)_p + 6\left(\frac{\partial v}{\partial y}\right)_p^3 + 3\left(\frac{\partial v}{\partial y}\right)_p\left(\frac{\partial^2 u}{\partial y^2}\right)_p,$$

$$\left(\frac{\partial F_1}{\partial z}\right)_p = \left(\frac{\partial F_2}{\partial z}\right)_p = \left(\frac{\partial^2 F_2}{\partial x\partial z}\right)_p = \left(\frac{\partial^2 F_2}{\partial y\partial z}\right)_p = 0.$$

We choose a pair of vector fields and calculate derivatives with respect to them as follows. Let η_1 and η_2 be C^∞ vector fields on U as

$$\eta_1 = \frac{\partial u}{\partial y}\frac{\partial}{\partial x} - \frac{\partial u}{\partial x}\frac{\partial}{\partial y},$$

$$\eta_2 = \frac{\partial}{\partial z}.$$

Noting that the coefficients of η_1 are derived functions,

$$\eta_1 F_2 = \frac{\partial u}{\partial y}\frac{\partial F_2}{\partial x} - \frac{\partial u}{\partial x}\frac{\partial F_2}{\partial y},$$

$$\begin{aligned}
\frac{\partial}{\partial x}\eta_1 F_2 &= \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\frac{\partial F_2}{\partial x} - \frac{\partial u}{\partial x}\frac{\partial F_2}{\partial y}\right)\\
&= \frac{\partial^2 u}{\partial x \partial y}\frac{\partial F_2}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial^2 F_2}{\partial x^2} - \frac{\partial^2 u}{\partial x^2}\frac{\partial F_2}{\partial y} - \frac{\partial u}{\partial x}\frac{\partial^2 F_2}{\partial x \partial y},\\
\frac{\partial}{\partial y}\eta_1 F_2 &= \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\frac{\partial F_2}{\partial x} - \frac{\partial u}{\partial x}\frac{\partial F_2}{\partial y}\right)\\
&= \frac{\partial^2 u}{\partial y^2}\frac{\partial F_2}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial^2 F_2}{\partial x \partial y} - \frac{\partial^2 u}{\partial x \partial y}\frac{\partial F_2}{\partial y} - \frac{\partial u}{\partial x}\frac{\partial^2 F_2}{\partial y^2},
\end{aligned}$$

$$\begin{aligned}
\eta_1\eta_1 F_2 &= \left(\frac{\partial u}{\partial y}\frac{\partial}{\partial x} - \frac{\partial u}{\partial x}\frac{\partial}{\partial y}\right)\eta_1 F_2\\
&= \frac{\partial u}{\partial y}\left(\frac{\partial}{\partial x}\eta_1 F_2\right) - \frac{\partial u}{\partial x}\left(\frac{\partial}{\partial y}\eta_1 F_2\right)\\
&= \frac{\partial u}{\partial y}\left(\frac{\partial^2 u}{\partial x \partial y}\frac{\partial F_2}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial^2 F_2}{\partial x^2} - \frac{\partial^2 u}{\partial x^2}\frac{\partial F_2}{\partial y} - \frac{\partial u}{\partial x}\frac{\partial^2 F_2}{\partial x \partial y}\right)\\
&\quad - \frac{\partial u}{\partial x}\left(\frac{\partial^2 u}{\partial y^2}\frac{\partial F_2}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial^2 F_2}{\partial x \partial y} - \frac{\partial^2 u}{\partial x \partial y}\frac{\partial F_2}{\partial y} - \frac{\partial u}{\partial x}\frac{\partial^2 F_2}{\partial y^2}\right)\\
&= \left(\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial x \partial y} - \frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial y^2}\right)\frac{\partial F_2}{\partial x} - \left(\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x \partial y}\right)\frac{\partial F_2}{\partial y}\\
&\quad + \left(\frac{\partial u}{\partial y}\right)^2\frac{\partial^2 F_2}{\partial x^2} + \left(\frac{\partial u}{\partial x}\right)^2\frac{\partial^2 F_2}{\partial y^2} - 2\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}\frac{\partial^2 F_2}{\partial x \partial y},
\end{aligned}$$

$$\begin{aligned}
&\eta_1\eta_1\eta_1 F_2\\
&=\left(\frac{\partial u}{\partial y}\frac{\partial}{\partial x}-\frac{\partial u}{\partial x}\frac{\partial}{\partial y}\right)\left(\left(\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial x\partial y}-\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial y^2}\right)\frac{\partial F_2}{\partial x}-\left(\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial x^2}-\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x\partial y}\right)\frac{\partial F_2}{\partial y}\right.\\
&\quad\left.+\left(\frac{\partial u}{\partial y}\right)^2\frac{\partial^2 F_2}{\partial x^2}+\left(\frac{\partial u}{\partial x}\right)^2\frac{\partial^2 F_2}{\partial y^2}-2\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}\frac{\partial^2 F_2}{\partial x\partial y}\right)\\
&=\left(\left(\frac{\partial u}{\partial x}\right)^2\frac{\partial^3 u}{\partial y^3}-2\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}\frac{\partial^3 u}{\partial x\partial y^2}+\left(\frac{\partial u}{\partial y}\right)^2\frac{\partial^3 u}{\partial x^2\partial y}-\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial x^2}\frac{\partial^2 u}{\partial y^2}+\frac{\partial u}{\partial y}\left(\frac{\partial^2 u}{\partial x\partial y}\right)^2\right)\frac{\partial F_2}{\partial x}\\
&\quad-\left(\left(\frac{\partial u}{\partial x}\right)^2\frac{\partial^3 u}{\partial x\partial y^2}-2\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}\frac{\partial^3 u}{\partial x^2\partial y}-\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2}\frac{\partial^2 u}{\partial y^2}+\frac{\partial u}{\partial x}\left(\frac{\partial^2 u}{\partial x\partial y}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2\frac{\partial^3 u}{\partial x^3}\right)\frac{\partial F_2}{\partial y}\\
&\quad+3\left(-\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial y^2}+\left(\frac{\partial u}{\partial y}\right)^2\frac{\partial^2 u}{\partial x\partial y}\right)\frac{\partial^2 F_2}{\partial x^2}+3\left(-\left(\frac{\partial u}{\partial x}\right)^2\frac{\partial^2 u}{\partial x\partial y}+\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial x^2}\right)\frac{\partial^2 F_2}{\partial y^2}\\
&\quad+3\left(\left(\frac{\partial u}{\partial x}\right)^2\frac{\partial^2 u}{\partial y^2}-\left(\frac{\partial u}{\partial y}\right)^2\frac{\partial^2 u}{\partial x^2}\right)\frac{\partial^2 F_2}{\partial x\partial y}+\left(\frac{\partial u}{\partial y}\right)^3\frac{\partial^3 F_2}{\partial x^3}-\left(\frac{\partial u}{\partial x}\right)^3\frac{\partial^3 F_2}{\partial y^3}\\
&\quad-3\frac{\partial u}{\partial x}\left(\frac{\partial u}{\partial y}\right)^2\frac{\partial^3 F_2}{\partial x^2\partial y}+3\left(\frac{\partial u}{\partial x}\right)^2\frac{\partial u}{\partial y}\frac{\partial^3 F_2}{\partial x\partial y^2}.
\end{aligned}$$

By the results of the previous paragraph,

$$\begin{aligned}
(\eta_1 F_2)_q&=\left(\frac{\partial u}{\partial y}\right)_q\left(\frac{\partial F_2}{\partial x}\right)_q-\left(\frac{\partial u}{\partial x}\right)_q\left(\frac{\partial F_2}{\partial y}\right)_q\\
&=\left(\frac{\partial u}{\partial y}\right)_q v_q\left(\frac{\partial u}{\partial x}\right)_q-\left(\frac{\partial u}{\partial x}\right)_q v_q\left(\frac{\partial u}{\partial y}\right)_q=0,
\end{aligned}$$

$$\begin{aligned}
\left(\frac{\partial}{\partial x}\eta_1 F_2\right)_p&=\left(\frac{\partial^2 u}{\partial x\partial y}\right)_p\left(\frac{\partial F_2}{\partial x}\right)_p+\left(\frac{\partial u}{\partial y}\right)_p\left(\frac{\partial^2 F_2}{\partial x^2}\right)_p\\
&\quad-\left(\frac{\partial^2 u}{\partial x^2}\right)_p\left(\frac{\partial F_2}{\partial y}\right)_p-\left(\frac{\partial u}{\partial x}\right)_p\left(\frac{\partial^2 F_2}{\partial x\partial y}\right)_p\\
&=2\left(\frac{\partial u}{\partial y}\right)_p\left(\frac{\partial u}{\partial x}\right)_p\left(\frac{\partial v}{\partial x}\right)_p-\left(\frac{\partial u}{\partial x}\right)_p\left(\left(\frac{\partial u}{\partial x}\right)_p\left(\frac{\partial v}{\partial y}\right)_p+\left(\frac{\partial u}{\partial y}\right)_p\left(\frac{\partial v}{\partial x}\right)_p\right)\\
&=-\left(\frac{\partial u}{\partial x}\right)_p\left(\left(\frac{\partial u}{\partial x}\right)_p\left(\frac{\partial v}{\partial y}\right)_p-\left(\frac{\partial u}{\partial y}\right)_p\left(\frac{\partial v}{\partial x}\right)_p\right),
\end{aligned}$$

$$\begin{aligned}
(\eta_1\eta_1F_2)_p &= \left(\left(\frac{\partial u}{\partial y}\right)_p\left(\frac{\partial^2 u}{\partial x\partial y}\right)_p - \left(\frac{\partial u}{\partial x}\right)_p\left(\frac{\partial^2 u}{\partial y^2}\right)_p\right)\left(\frac{\partial F_2}{\partial x}\right)_p \\
&\quad - \left(\left(\frac{\partial u}{\partial y}\right)_p\left(\frac{\partial^2 u}{\partial x^2}\right)_p - \left(\frac{\partial u}{\partial x}\right)_p\left(\frac{\partial^2 u}{\partial x\partial y}\right)_p\right)\left(\frac{\partial F_2}{\partial y}\right)_p \\
&\quad + \left(\frac{\partial u}{\partial y}\right)_p^2\left(\frac{\partial^2 F_2}{\partial x^2}\right)_p - 2\left(\frac{\partial u}{\partial x}\right)_p\left(\frac{\partial u}{\partial y}\right)_p\left(\frac{\partial^2 F_2}{\partial x\partial y}\right)_p + \left(\frac{\partial u}{\partial x}\right)_p^2\left(\frac{\partial^2 F_2}{\partial y^2}\right)_p \\
&= 2\left(\frac{\partial u}{\partial y}\right)_p^2\left(\frac{\partial u}{\partial x}\right)_p\left(\frac{\partial v}{\partial x}\right)_p + 2\left(\frac{\partial u}{\partial x}\right)_p^2\left(\frac{\partial u}{\partial y}\right)_p\left(\frac{\partial v}{\partial y}\right)_p \\
&\quad - 2\left(\frac{\partial u}{\partial x}\right)_p\left(\frac{\partial u}{\partial y}\right)_p\left(\left(\frac{\partial u}{\partial x}\right)_p\left(\frac{\partial v}{\partial y}\right)_p + \left(\frac{\partial u}{\partial y}\right)_p\left(\frac{\partial v}{\partial x}\right)_p\right) \\
&= 0,
\end{aligned}$$

and similarly,

$$\left(\frac{\partial}{\partial y}\eta_1F_2\right)_p = -\left(\frac{\partial u}{\partial y}\right)_p\left(\left(\frac{\partial u}{\partial x}\right)_p\left(\frac{\partial v}{\partial y}\right)_p - \left(\frac{\partial u}{\partial y}\right)_p\left(\frac{\partial v}{\partial x}\right)_p\right),$$

$$(\eta_1\eta_1\eta_1F_2)_p = -6\left(\left(\frac{\partial u}{\partial x}\right)_p\left(\frac{\partial v}{\partial y}\right)_p - \left(\frac{\partial u}{\partial y}\right)_p\left(\frac{\partial v}{\partial x}\right)_p\right)^3 \neq 0.$$

Similarly, we can obtain that $(\eta_1F_1)_q = (\eta_2F_1)_p = (\eta_2F_2)_p = (\eta_1\eta_2F_2)_p = (\eta_2\eta_1F_2)_p = 0$ and

$$(\eta_2\eta_2F_2)_p = \left(\frac{\partial^2 F_2}{\partial z^2}\right)_p = 2\left(\frac{\partial w}{\partial z}\right)_p > 0.$$

We are now ready to complete the proof. By the above, $(dF_1)_p \neq (0,0,0)$ and $(dF_2)_p = (0,0,0)$. Since $(\eta_1)_p$ and $(\eta_2)_p$ are linearly independent, and $(\eta_1F_1)_p = (\eta_2F_1)_p = (\eta_1F_2)_p = (\eta_2F_2)_p = 0$, we obtain the condition that $\ker(dF)_p = \left\langle(\eta_1)_p, (\eta_2)_p\right\rangle$. Since $(\eta_1F_1)_q = (\eta_1F_2)_q = 0$, we obtain the condition that $(\eta_1)_q \in \ker(dF)_q$ for $q \in S(F)\cap U$. The matrix

$$\begin{pmatrix} (\eta_1\eta_1F_2)_p & (\eta_1\eta_2F_2)_p \\ (\eta_2\eta_1F_2)_p & (\eta_2\eta_2F_2)_p \end{pmatrix},$$

denoted by $(\mathbf{H}_{\eta_1,\eta_2}F_2)_p$, is equal to

$$\begin{pmatrix} 0 & 0 \\ 0 & 2\left(\frac{\partial w}{\partial z}\right)_p \end{pmatrix},$$

which shows that $\ker(\mathbf{H}_{\eta_1,\eta_2}F_2)_p = \left\langle(\eta_1)_p\right\rangle$. Since

$$\left(\left(\frac{\partial}{\partial x}\eta_1F_2\right)_p, \left(\frac{\partial}{\partial y}\eta_1F_2\right)_p\right)$$

$$= -\left(\left(\frac{\partial u}{\partial x}\right)_p\left(\frac{\partial v}{\partial y}\right)_p - \left(\frac{\partial u}{\partial y}\right)_p\left(\frac{\partial v}{\partial x}\right)_p\right)\left(\left(\frac{\partial u}{\partial x}\right)_p, \left(\frac{\partial u}{\partial y}\right)_p\right) \neq (0,0),$$

we obtain the condition that $(d(\eta_1 F_2))_p \neq (0,0,0)$, as well as $(\eta_1\eta_1\eta_1 F_2)_p \neq 0$. By Theorem 3, the point p is a cusp point of F.

3.4 $\Sigma^{2,0}_{1,0}$ case

The last assertion of the proposition can be proved by a simple observation as follows. Suppose that p is a $\Sigma^{2,0}_{1,0}$ point of f. Let $S(f)$ and $S(F)$ denote the singular sets of f and F, respectively, $S\left(f|_{\partial\mathbb{R}^3_{\geq 0}}\right)$ denote the boundary singular set of f, and U be a sufficiently small neighborhood of p in $\mathbb{R}^3$. Recall that $\left(S(f) \cup S\left(f|_{\partial\mathbb{R}^3_{\geq 0}}\right)\right) \cap U$ is a figure $\perp$ consisting only of fold points, boundary fold points and the $\Sigma^{2,0}_{1,0}$ point p. By the results of the previous subsections, $S(F) \cap U$ is a figure $+$ where the crossing point is p. This shows that p is neither a regular point, a fold point nor a cusp point of F.

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